$$-(1/\pi)\int_0^\pi \sum_{m=1}^{N} m\, a_m \frac{\sin m\theta}{\sin\theta} \cdot \frac{\sum_{n=1}^{N} n\, a_n \cos n\theta}{(\cos\theta-\cos\phi)^2}\, d\theta$$

$$= \sum_{m=1}^{N} m\, a_m \frac{\sin m\phi}{\sin\phi} \sum_{n=1}^{N} n\, a_n I_{1,n}$$

$$- \sum_{m=1}^{N} \sum_{n=1}^{N} m\, n\, a_m\, a_n\, I_{m,n} \tag{8a}$$

where

$$I_{m,n} = (1/\pi)\int_0^\pi \frac{\sin m\theta \cos n\theta}{\sin\theta\,(\cos\theta-\cos\phi)^2}\, d\theta \tag{8b}$$

If the following trigonometric identities are used

$$\frac{\sin m\theta}{\sin\theta} = 2 \sum_{\substack{k=\xi+1\\ \xi+3,\ldots}}^{m-1} \cos k\theta + \xi \tag{9}$$

$$2\sin k\theta \sin n\theta = \cos(k-n)\theta - \cos(k+n)\theta \tag{10a}$$

$$2\cos k\theta \cos n\theta = \cos(k-n)\theta + \cos(k+n)\theta \tag{10b}$$

$$2\sin k\theta \cos n\theta = \sin(k-n)\theta + \sin(k+n)\theta \tag{10c}$$

where

$$\xi = 0 \text{ for even } m \tag{11a}$$

$$= 1 \text{ for odd } m \tag{11b}$$

then

$$I_{m,n} = -(1/\sin^3\phi) \sum_{\substack{k=\xi+1\\ \xi+3,\ldots}}^{m-1}$$

$$\times [(k+n)\cos(k+n)\phi \sin\phi - \sin(k+n)\phi\cos\phi$$

$$+ |(k-n)| \cos(k-n)\phi \sin\phi - \sin|(k-n)|\phi \cos\phi] + \xi I_{1,n} \tag{12}$$

$$I_{1,n} = -(1/\sin^3\phi)[n\cos n\phi \sin\phi - \sin n\phi \cos\phi] \tag{13}$$

However, in numerical computations one may prefer to use the following recursive relation for $I_{m,n}$

$$I_{m,n} = \tfrac{1}{2}[I_{m-1,\,n+1} + I_{m-1,\,n-1} + I_{1,\,m+n-1} + I_{1,\,m-n-1}] \tag{14}$$

With all the $I_{m,n}$ known, the inplane acceleration can be calculated from Eq. (8).

These results, with Yates' assumption concerning multiple vortex roll-ups, may be used to obtain the number and strengths of the rolled-up vortex cores, for example, by using Rossow's analysis,[3] to determine aircraft wake hazard potential.

As an interesting example, consider an aircraft in symmetric flight carrying an elliptic span loading. Then it can be inferred that at $t=0$, the downwash in the wake is uniform and that the inplane acceleration is zero. This implies that the wake roll-up process is likely to be slow and hence the wake hazard may persist over larger distances. However, this loading gives a near uniform downwash field at the horizontal tail which could be a desirable feature from a stability and control point of view, as compared to multiple vortex cores forming in the wake and passing close to the horizontal tail. Multiple vortex cores have been suggested as a possible method of alleviating wake hazard potential by having several pronounced dips in the spanwise loading. Whereas this could be a useful solution during a landing approach, it is likely to give unacceptable drag penalities during take-off and climb, particularly when engine noise suppression measures have already started biting into available thrust.

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Derivatives of a Statically Reduced Stiffness Matrix with Respect to Sizing Variables

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An expression is derived for the first derivatives with respect to the sizing variables of a statically reduced stiffness matrix that is a nonlinear function of the sizing variables, where the unreduced stiffness matrix is a linear function of the sizing variables. In most methods of structural optimization (weight minimization) with flutter constraints the derivatives of the structural stiffness matrix with respect to the sizing variables (or design variables) β_i are used.[1] A structural representation satisfactory for aeroelastic analyses can be obtained from a finite element approach in which the elements are chosen such that the total stiffness matrix $[K(\beta_i)]$ is a linear function of the sizing variables:

$$[K(\beta_i)] = [K_0] + \sum_{i=1}^{n} \beta_i [\Delta K_i] \tag{1}$$

where $[K_0]$ represents an invariable stiffness corresponding to $[K(0)]$ and $[\Delta K_i]$ is a stiffness matrix associated with the sizing variable β_i. Under this condition the derivative of the stiffness matrix with respect to any sizing variable is a matrix that is independent of the sizing variables:

$$\frac{\partial}{\partial\beta_i}[K(\beta_i] = [\Delta K_i] \tag{2}$$

Thus during an optimization procedure consisting of several steps, for each of which $[K(\beta_i)]$ must be evaluated according to Eq. (1), the derivatives of the stiffness matrix need to be evaluated only once.

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In a realistic design environment the number of nodal displacements may become too large for use as degrees of freedom in an optimization procedure with flutter constraints. An accepted procedure to reduce the number of degrees of freedom is to eliminate a number of nodal displacements from the degrees of freedom such that the accuracy of the flutter analysis is not significantly affected. A common approach is one in which nodal displacements to be eliminated are assumed to have zero loads. This is often called static reduction in contrast with the approach of Ref. 2, which is called dynamic reduction, since the reduction is a function of the mass and the frequency.

Let

$$[K(\beta_i)] = \left[\begin{array}{c|c} K_{11}(\beta_i) & K_{12}(\beta_i) \\ \hline K_{21}(\beta_i) & K_{22}(\beta_i) \end{array}\right] \tag{3}$$

where the subscript 1 refers to degrees of freedom to be retained and subscript 2 to degrees of freedom to be eliminated. If $\{Z\}$ represents loads and $\{z\}$ displacements, Eq. (3) implies the following relation:

$$\begin{Bmatrix} Z_1 \\ Z_2 \end{Bmatrix} = \left[\begin{array}{c|c} K_{11}(\beta_i) & K_{12}(\beta_i) \\ \hline K_{21}(\beta_i) & K_{22}(\beta_i) \end{array}\right] \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} \tag{4}$$

Assuming $Z_2 = 0$ leads to

$$0 = [K_{21}(\beta_i)]\ \{z_1\} + [K_{22}(\beta_i)\ ]\ \{z_2\} \tag{5}$$

from which

$$\{z_2\} = -[K_{22}(\beta_i)]^{-1}\ [K_{21}(\beta_i)]\ \{z_1\} \tag{6}$$

Substituting Eq. (6) in the upper half of Eq. (4) leads to

$$\{Z_1\} = \Big[\ [K_{11}(\beta_i)] - [K_{12}(\beta_i)]\ [K_{22}(\beta_i)]^{-1} \times [K_{21}(\beta_i]\ \Big]\{z_1\} \tag{7}$$

which defines a reduced stiffness matrix $[RK(\beta_i)]$ as

$$[RK(\beta_i)] = [K_{11}(\beta_i)] - [K_{12}(\beta_i] \times [K_{22}(\beta_i)]^{-1}[K_{21}(\beta_i)] \tag{8}$$

The second term on the right hand side of Eq. (8) causes $[RK(\beta_i]$, in the general case, to be a nonlinear function of β_i. Thus the derivatives of the reduced stiffness matrix are functions of β_i and must be evaluated several times during an optimization procedure.

For the following it is convenient to use an alternate expression for $[RK(\beta_i)]$

$$[RK(\beta_i)] = \Big[\ [I]\ \Big|\ [G^T(\beta_i)]\ \Big][K(\beta_i)] \left[\begin{array}{c} [I] \\ \hline [G(\beta_i)] \end{array}\right] \tag{9}$$

where

$$[G(\beta_i)] = -[K_{22}(\beta_i)]^{-1}\ [K_{21}(\beta_i)] \tag{10}$$

and the superscript T indicates a matrix transpose. The matrix

$$[GR(\beta_i)] = \left[\begin{array}{c} [I] \\ \hline [G(\beta_i)] \end{array}\right] \tag{11}$$

is sometimes referred to as the Guyan reduction matrix.[3] It should be noted that by definition

$$[K(\beta_i)]\ [GR(\beta_i)] = \left[\begin{array}{c} [RK(\beta_i)] \\ \hline [0] \end{array}\right] \tag{12}$$

The derivative of the reduced stiffness matrix with respect to a design variable is given by:

$$\frac{\partial}{\partial\beta_i}[RK(\beta_i)] = \left[\frac{\partial}{\partial\beta_i}GR^T(\beta_i)\right][K(\beta_i)][GR(\beta_i)] + [GR^T(\beta_i)]\left[\frac{\partial}{\partial\beta_i}K(\beta_i)\right][GR(\beta_i)] + [GR^T(\beta_i)]\ [K(\beta_i)]\left[\frac{\partial}{\partial\beta_i}GR(\beta_i)\right] \tag{13}$$

From Eq. (11) it follows:

$$\left[\frac{\partial}{\partial\beta_i}GR^T(\beta_i)\right] = \left[[0]\ \Big|\ \frac{\partial}{\partial\beta_i}[G(\beta_i)]\right] \tag{14}$$

Combining Eq. (14) with Eq. (12) leads to:

$$\left[\frac{\partial}{\partial\beta_i}GR^T(\beta_i)\right][K(\beta_i)]\ [GR(\beta_i)] = 0 \tag{15}$$

Similarly it can be shown that the third term on the right hand side of Eq. (13) is zero. Thus, with the help of Eq. (2) it is found:

$$\frac{\partial}{\partial\beta_i}[RK(\beta_i)] = [GR^T(\beta_i)]\ [\Delta K_i]\ [GR(\beta_i)] \tag{16}$$

Eq. (16) shows that one set of $[\Delta K_i]$ can be used to evaluate all the derivatives of the reduced stiffness matrix at all sets of β_i values. For each set of β_i values a new matrix $[GR(\beta_i)]$ must be determined.

In a typical optimization procedure with flutter constraints, the derivative of the stiffness matrix may be used in the following manner[1]

$$\lfloor r \rfloor \left[\frac{\partial}{\partial\beta_i}RK(\beta_i)\right]\{q\} = \lfloor r \rfloor\ [GR^T(\beta_i)]\ [\Delta K_i]\ [GR(\beta_i)]\ \{q\} \tag{17}$$

where $\{q\}$ is the characteristic vector of the flutter matrix equation and $\lfloor r \rfloor$ the transpose of the characteristic vector of the adjoint flutter matrix equation corresponding to a particular solution of the flutter equation. For each step in an optimization procedure one $[GR(\beta_i)]$, $\{q\}$ and $[r]$ are defined and thus it is computationally advantageous to first perform the multiplications $[GR(\beta_i)]\ \{q\}$ and $\lfloor r \rfloor\ [GR^T(\beta_i)]$, and then evaluate Eq.(17) for all $[\Delta K_i]$.

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